

SPECTRAL ESTIMATORS FOR MULTI-INDEX MODELS: PRECISE ASYMPTOTICS AND OPTIMAL WEAK RECOVERY

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Setup

Multi-Index Model

$$\underset{\mathbb{R}^d}{\mathbf{x}} \xrightarrow{W^*} \underset{\mathbb{R}^p}{W^* \mathbf{x}} \xrightarrow{q(\cdot) + \varepsilon} \underset{\mathbb{R}}{\mathbf{y}} = q(W^* \mathbf{x}) + \varepsilon$$

$q: \mathbb{R}^p \rightarrow \mathbb{R}$ nonlinearity, $W^* \in \mathbb{R}^{p \times d}$ signal matrix, and ε random noise.

Remark Taking $q = (b_i \cdot \text{ReLU})^{\otimes p}$ gives a two-layer neural network.

Weak Recovery

$$\widehat{W} = [\hat{w}_1, \dots, \hat{w}_p]^\top \text{ weakly recovers } W^* = [w_1^*, \dots, w_p^*]^\top \text{ iff } \exists i, j \in [p] \text{ such that } \frac{|\langle \hat{w}_i, w_j^* \rangle|}{\|\hat{w}_i\|_2 \|w_j^*\|_2} \xrightarrow{d \rightarrow \infty} 0.$$

Spectral estimator

Fix any $\mathcal{T}: \mathbb{R} \rightarrow \mathbb{R}$. Define the matrix $D \equiv D_n \in \mathbb{R}^{d \times d}$ as

$$D_n = \frac{1}{n} X \mathcal{T}(Y) X^\top = \frac{1}{n} \sum_{i=1}^n \mathcal{T}(y_i) x_i x_i^\top.$$

The **spectral estimator** outputs top p eigenvectors of D , i.e.

$$\widehat{W} = [v_1^D, \dots, v_p^D].$$

Problem Formulation

Given $X \in \mathbb{R}^{d \times n}$, $Y \in \mathbb{R}^n$, q , and $W^* W^{*\top}$, weakly recover $W^* \in \mathbb{R}^{d \times p}$ for $Y = q(W^* X) + \varepsilon$.

Assumptions

- Gaussian data

$$X \in \mathbb{R}^{d \times n}, \text{ s.t. } X_{i,j} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1).$$

- Independent signals

$(w_i^*)_{i \in [p]}$ are linearly independent, and independent of X

- Proportional scaling, fixed index

$$n, d \rightarrow \infty, \frac{n}{d} = \delta, p = \Theta(1).$$

Optimal recovery

Theorem 1

There exists a weak recovery threshold δ_c such that

- For any $\delta \leq \delta_c$, **no** $\mathcal{T}$ can achieve weak recovery.
- For any $\delta > \delta_c$, **there exists** $\mathcal{T}_\delta^*$ that achieves weak recovery.

Both δ_c and $\mathcal{T}_\delta^*$ can be computed

$$\delta_c = \left[\max_{u \in \mathbb{S}^{p-1}} \int_{\mathbb{R}} \frac{(\mathbb{E}_s[p(y|s) \cdot \langle s, u \rangle^2 - 1])^2}{\mathbb{E}_s[p(y|s)]} dy \right]^{-1}, \quad (1)$$

$$\mathcal{T}^*(y) := 1 - \frac{\mathbb{E}_s[p(y|s)]}{\mathbb{E}_s[p(y|s) \cdot \langle s, u_c \rangle^2]}, \quad \mathcal{T}_\delta^*(y) := \frac{\sqrt{\delta_c} \cdot \mathcal{T}^*(y)}{\sqrt{\delta} - (\sqrt{\delta} - \sqrt{\delta_c}) \cdot \mathcal{T}^*(y)},$$

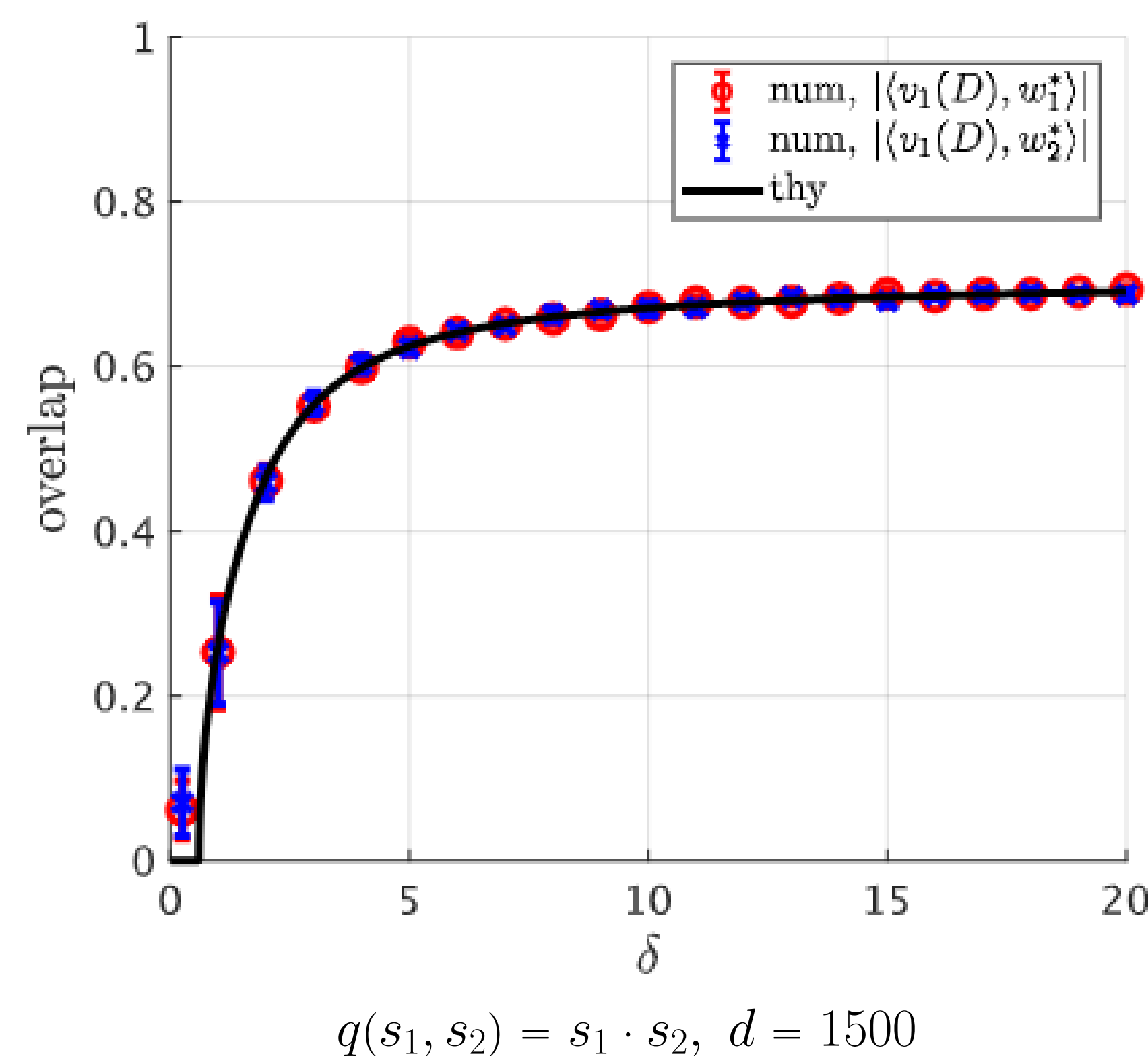
for $u_c \in \mathbb{S}^{p-1}$ a maximizer of (1).

Example

For $p = 2$, $q(s_1, s_2) = s_1 \cdot s_2$

$$\delta_c \approx 0.5937, \quad \mathcal{T}^*(y) = 1 - \frac{K_0(|y|)}{y K_0(|y|) + |y| K_1(|y|)}.$$

Simulation



Precise asymptotics

Fix $\mathcal{T}$, and $\tau = \inf\{c: \mathbb{P}(\mathcal{T}(y) \leq c) = 1\}$. Define $\psi_\delta:]\tau, \infty[\rightarrow \mathbb{R}$ and its point of minimum as

$$\psi_\delta(\alpha) := \alpha \left(\frac{1}{\delta} + \mathbb{E} \left[\frac{z}{\alpha - z} \right] \right), \quad \bar{\alpha}_\delta := \underset{\alpha > \tau}{\operatorname{argmin}} \psi_\delta(\alpha).$$

Define $\zeta_\delta: \mathbb{R} \rightarrow \mathbb{R}$ and $R^\infty:]\tau, \infty[\rightarrow \mathbb{R}^{p \times p}$ as

$$\zeta_\delta(\alpha) := \psi_\delta(\max\{\bar{\alpha}_\delta, \alpha\}), \quad R^\infty(\alpha) := \mathbb{E} \left[\frac{\alpha s s^\top z}{\alpha - z} \right],$$

where $s \sim \mathcal{N}(0_p, I_p)$, $z \sim \mathcal{T}(y)$. Let h_i^∞ be i -th eigenvector of $R^\infty(\alpha)$.

Theorem 2

Suppose $\alpha_i > \bar{\alpha}_\delta$ solution exists to

$$\lambda_i(R^\infty(\alpha)) = \zeta_\delta(\alpha).$$

Then, it holds that

$$\lambda_i^D \xrightarrow{a.s.} \zeta_\delta(\alpha_i),$$

$$\langle v_i^D, e_j \rangle \xrightarrow{a.s.} \frac{\zeta'_\delta(\alpha_i) |\langle h_i^\infty, e_j \rangle|^2}{\zeta'_\delta(\alpha_i) + h_i^{\infty\top} \frac{d}{d\alpha} R^\infty(\alpha_i) h_i^\infty}.$$

Otherwise, if no $\alpha_i > \bar{\alpha}_\delta$ solves it, then

$$\lambda_i^D \xrightarrow{a.s.} \zeta_\delta(\bar{\alpha}_\delta),$$

$$\langle v_i^D, e_j \rangle \xrightarrow{a.s.} 0.$$

Remark This builds on top of the results for $p = 1$ from (Mondelli, and Montanari, 2019).

Intuition

